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CARTEL ENFORCEMENT WITH UNCERTAINTY ABOUT COSTS

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by

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Abstract

What cartel agreements are possible when firms have private information about production costs? In order for a cartel agreement to work it must take into account the incentives for firms to misrepresent their cost information and it must provide sufficient reward so that no firm wishes to defect. For *private cost uncertainty* we characterize the set of cartel agreements with side payments that can be supported as Bayesian Nash equilibria. We show that if defection results in either Cournot or Bertrand competition the incentive problems in large cartels are severe enough to prevent the cartel from achieving the monopoly outcome. In contrast, with *common cost uncertainty*, the incentive problems become less severe in large cartels, allowing perfect collusion.

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1. *Introduction*

Several firms are interested in forming a cartel in the hopes of improving profitability within the industry. Although the output of every firm is jointly observed, each firm has private information about its own production costs.[1] For an industry with linear demand and linear production costs, we characterize the set of cartel agreements (called enforceable cartels) that can be supported by a Bayesian Nash equilibrium. We assume that side-payments among firms are possible. Thus, a low-cost firm is able to bribe a high-cost firm into not producing. The bribery, however, is complicated by the incomplete information, since the firms may have an incentive to misrepresent their costs. A firm that feels it is likely that it has the lowest cost in the industry may be tempted to understate the size of the industry profits by overstating its costs, whereas a firm with higher costs prefers to understate its costs so as to overstate its willingness to produce.

The set of enforceable cartels depends in a crucial way on what happens if the firms cannot agree on an allocation mechanism. We assume that if one firm refuses to join the cartel, the cartel breaks down resulting in either Cournot or Bertrand competition among all the firms in the industry. These outcomes in the face of disagreement determine each firm's individual rationality constraint. This is a relatively weak form of individual rationality, since it assumes the firms are able to commit to relatively aggressive noncooperative behavior if one firm refuses to participate.

Alternatively, one might consider a stronger form of individual rationality in which if one firm refuses to join, the $n - 1$ remaining firms

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[1]Related studies with uncertainty about production costs have focused primarily on regulatory issues. See, for example, Baron and Besanko [1984], Baron and Myerson [1982], Laffont and Tirole [1984], Loeb and Magat [1979], and Riordan and Sappington [1985].

continue as a cooperative cartel so that the industry looks more like a duopoly. Clearly, the less competitive is the behavior of the remaining $n - 1$ firms, the stronger the individual rationality constraint becomes. Thus, if we instead assumed the $n - 1$ remaining firms adopted the joint profit maximizing outcome in the face of a deviating firm, then the individual rationality constraint would be more severe and the set of enforceable cartels would be greatly reduced.

With complete information it is easy to sustain the monopoly outcome via an enforceable cartel—simply have the lowest-cost firm produce the monopoly output and pay the other firms at least their Cournot or Bertrand profits. But with incomplete information, the monopoly outcome may not be enforceable due to the firms' incentives to misrepresent their true costs. Our main result is that when there are sufficiently many firms in the industry, the monopoly outcome is unobtainable, even with the weak individual-rationality constraint given by our Cournot or Bertrand assumption. This result is in agreement with conventional wisdom that suggests as the number of firms grows, it becomes increasingly difficult to maintain the cartel.

In our model, the difficulty arises as a result of asymmetric information. Simply put, with more firms, a greater amount of surplus is needed, since both producing and nonproducing firms have to be subsidized. We require that the aggregate subsidy cannot exceed the total (ex post) revenue generated by the cartel.

Although intuitive, the result does have some bite, since aside from the incomplete information about costs, our setting—symmetry, side-payments, full information about quantities and other data, and an ability to commit to an allocation mechanism—tends to favor efficient cartels. Hence, even when it is possible for the firms to establish binding agreements with side-payments, the firms' incentives to misrepresent production costs precludes the formation of a cartel that attains the monopoly outcome.

Finally, we consider the effect of common cost uncertainty on the ability for a group of firms to successfully collude. By "common" we mean that all firms have identical but unknown cost functions and each firm has a private signal which is informative about these costs. For this case we do not characterize the optimal mechanism in general. Instead, we find a specific mechanism which achieves the monopoly outcome if there are enough firms. This result is established for a very broad class of disagreement games, specifically, all games which satisfy a particular uniformity property. This contrasts sharply with the results for private cost uncertainty.

Cartel agreements of the sort analyzed here are typically illegal in a domestic industry, because of their negative welfare affects. However, in export industries, governments would tend to encourage cartel arrangements, since the exporting nations are solely interested in maximizing producer surplus. Therefore, we might expect some degree of cartel formation involving side-payments to exist in export industries with few firms, such as precious commodity production (diamonds and platinum), or in countries specializing in the export of a single agricultural commodity.

Before presenting our model, we want to emphasize that there are many alternative ways to pose the cartel problem when there is asymmetric information. A few possibilities have been mentioned above (side-payments vs. no side-payments; different assumptions about what happens when the cartel falls apart—leading to different specifications of the individual rationality constraints; the degree to which the cartel members are required to commit to the rules of the cartel).[2] What we have tried to do is make assumptions which are *conducive* to the successful formation of efficient cartels. What we have shown is that, even within this favorable environment, if firms have private information about costs, a cartel cannot successfully enforce the monopoly outcome when there are many firms. We have, however, ignored dynamic aspects of cartel enforcement. A repeated game model of competition might well expand the set of enforceable cartels.[3]

We begin by formulating the model and analyzing the direct revelation game. Then in Section 3 the two alternative threats (Cournot or Bertrand competition) are studied. In Section 4, we give necessary and sufficient conditions for a cartel to attain the monopoly outcome, and in Section 5 the case of uniform uncertainty is worked out to demonstrate that the monopoly outcome is not attainable by an enforceable cartel when there are too many firms. Section 6 analyzes cartel enforcement for a model in which each firm has a private signal about a common industry production cost. Finally, in Section 7 we discuss some extensions and alternative formulations of our model.

2. The Revelation Game

An industry consists of n firms indexed by $i \in N = (1, \ldots, n)$. Each firm in the industry produces the quantity q_i of a homogeneous good and incurs linear production costs $c_i q_i$. The market price for the good depends

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[2] See Roberts[1983,1985] for an analysis of cartel enforcement without side payments.

[3] For dynamic studies of collusive behavior, see Abreu [1986], Abreu, Pearce, and Stacchetti [1986], Fudenberg and Maskin [1986], Green and Porter [1984], and Porter [1983].

on industry output $q = (q_1, \ldots, q_n)$ as given by the linear inverse demand $P(q) = a - \sum q_i$.[4] The output of each firm is jointly observable, but a firm's variable cost c_i is known privately. Other firms know only that the private cost parameter of firm i is drawn independently from the continuous distribution F, which has a positive density f on $[\underline{c},\bar{c}]$ with $c \le a$. Industry information, such as the demand schedule, the form of the cost function, and the distribution F, is common-knowledge.

In deriving the set of enforceable cartels, we consider the direct revelation game in which firms simultaneously report their variable costs $c = (c_1, \ldots, c_n)$, resulting in an allocation $q(c) = (q_1(c), \ldots, q_n(c))$ and $r(c) = (r_1(c), \ldots, r_n(c))$, where $q_i \ge 0$ is the output of firm i and r_i is firm i's share of the industry revenues for producing q_i.[5] The pair of outcome rules $\langle q,r\rangle$, which determine an allocation as a function of the firms' reports, is called a cartel. A *feasible cartel* satisfies the ex post budget balance constraint that the sum of the revenue shares equals the total industry revenue:

$$\text{(B)} \qquad \sum_{i=1}^{n} r_i(c) = \rho(c) \equiv \left[a - \sum_{i=1}^{n} q_i(c) \right] \sum_{i=1}^{n} q_i(c) \qquad \text{for all } c \in [\underline{c},\bar{c}]^n .$$

A firm seeks to maximize expected profit. Ex post a firm with cost c_i, producing the quantity q_i, and receiving the revenue r_i has a profit of

[4] None of the characterization results in this section depend on our assumption of a linear demand curve or the assumption of linear costs. What is important is that the private information enter linearly in the form $C_i(q_i) = c_i H(q_i)$, where $H(\cdot)$ is some continuous increasing function. Both assumptions, however, do play important roles in later sections.

[5] Our analysis of the direct revelation game draws heavily upon the prior work of Baron and Myerson [1982], Cramton, Gibbons, and Klemperer [1987], and Myerson and Satterthwaite [1983].

$r_i - c_i q_i$. Let $-i = N\backslash i$ and let $E_{-i}(\cdot)$ be the expectation operator with respect to c_{-i}. Then we can define the expected production and revenue for firm i when it announces c_i by

$$Q_i(c_i) = E_{-i}\{q_i(c)\} \qquad \text{and} \qquad R_i(c_i) = E_{-i}\{r_i(c)\},$$

so the firm's expected profit is

$$\pi_i(c_i) = R_i(c_i) - c_i Q_i(c_i).$$

Incentive Compatibility

A cartel $\langle q,r\rangle$ is *incentive compatible* if all types of all firms want to report their private information truthfully:

$$\pi_i(c_i) \geq R_i(v) - c_i Q_i(v) \qquad \text{for all } i \in N, \text{ and } c_i, v \in [\underline{c},\bar{c}].$$

By the *revelation principle*, we lose no generality by restricting attention to incentive compatible cartels.

The assumptions of independence and risk neutrality allow us to give a convenient representation of incentive compatibility. In particular, for any incentive compatible cartel, there is a precise relationship between the expected revenue R_i and the expected production Q_i: once q is specified both π_i and R_i are determined up to a constant.

Lemma 1. *The cartel $\langle q,r\rangle$ is incentive compatible if and only if for every $i \in N$, Q_i is decreasing and for all $c', c_i \in [\underline{c},\bar{c}]$*

$$(IC) \qquad R_i(c') - R_i(c_i) = \int_{c_i}^{c'} v\,dQ_i(v).$$

Moreover, if $\langle q,r\rangle$ is incentive compatible, then π_i is convex and decreasing with derivative $d\pi_i/dc_i = -Q_i$ almost everywhere and for all $c', c_i \in [\underline{c},\bar{c}]$

$$\pi_i(c') - \pi_i(c_i) = -\int_{c_i}^{c'} Q_i(v)dv.$$

Proof: *Only if*. If $\langle q,r\rangle$ is incentive compatible, then

$$\pi_i(c_i) = R_i(c_i) - c_i Q_i(c_i) \geq R_i(v) - c_i Q_i(v),$$

or equivalently

$$\pi_i(c_i) \geq \pi_i(v) - (c_i - v)Q_i(v),$$

implying that π_i has a supporting hyperplane at v with slope $-Q_i(v) \leq 0$. Thus, π_i is convex and has derivative $d\pi_i/dc_i = -Q_i$ almost everywhere. Also, Q_i must be decreasing, and

$$\pi_i(c') - \pi_i(c_i) = -\int_{c_i}^{c'} Q_i(v)dv.$$

(We use the Stieltjes integral throughout this paper, so that any discontinuities in the expected production function Q_i are accounted for in the integral.) By integration by parts,

$$\int_{c_i}^{c'} Q_i(v)dv = c'Q_i(c') - c_iQ_i(c_i) - \int_{c_i}^{c'} vdQ_i(v),$$

which together with the definition of π_i yields (IC).

If. Subtracting the identity

$$c_i[Q_i(c') - Q_i(c_i)] = c_i \int_{c_i}^{c'} dQ_i(c_i)$$

from (IC) results in

$$R_i(c') - R_i(c_i) - c_iQ_i(c') + c_iQ_i(c_i) = \int_{c_i}^{c'} (v - c_i)dQ_i(c_i) \leq 0,$$

where the inequality follows because the integrand is nonpositive for all c_i and $v \in [c_i, \bar{c}]$, since Q_i is decreasing. Rearranging the terms on the lefthand side yields

$$R_i(c_i) - c_iQ_i(c_i) \geq R_i(c') - c_iQ(c'),$$

which is incentive compatibility. ■

Individual Rationality

In order to define the set of enforceable cartels, it is necessary to specify exactly what happens if one firm refuses to join (or defects from) the cartel. We assume that a single defection from the proposed cartel by one of the firms leads to a complete break-down of the cartel and so the industry produces as an n-firm Cournot oligopoly with output determined by either Cournot or Bertrand competition. Let $\pi^{\alpha}(c)$ be the expected payoff to a firm with cost c if the cartel is not formed, where α = C or B depending on whether Cournot or Bertrand competition occurs when one or more firms refuse to participate in the cartel. Similarly, define $Q^{\alpha}(c)$ and $R^{\alpha}(c)$ to be the expected quantity and revenue of firm c if the cartel breaks down. The functions π^{α}, Q^{α}, and R^{α} are determined in the next section for both Cournot and Bertrand competition. These functions do not depend on i due to the symmetry of the model.

The cartel $\langle q,r\rangle$ is said to be *individually rational* with respect to the threat α if all firms are better-off joining the cartel than defecting:

$$\pi_i(c_i) \geq \pi^{\alpha}(c_i) \qquad \text{for all } i \in N \text{ and } c_i \in [\underline{c},\bar{c}].$$

A feasible cartel that is incentive compatible and individually rational is called an *enforceable cartel*, since the allocation implied by an enforceable cartel is the outcome of a Bayesian Nash equilibrium.

Lemma 2. *An incentive-compatible cartel $\langle q,r\rangle$ is individually rational if and only if for all $i \in N$ and $\hat{c}_i \in \hat{C}_i = \{\underline{c},\bar{c}\} \cup \{c_i \mid Q_i(c_i) = Q^{\alpha}(c_i)\}$*

$$(IR) \qquad R_i(\hat{c}_i) - \hat{c}_i Q_i(\hat{c}_i) \geq \pi^{\alpha}(\hat{c}_i).$$

Proof: It is necessary and sufficient to check individual rationality at costs which minimize a firm's net payoff $U_i^{\alpha}(c_i) = \pi_i(c_i) - \pi^{\alpha}(c_i)$, since if individual rationality is satisfied for the worst-off types of firms it is satisfied for all types. The continuity of both π_i and π^{α} implies that $U_i^{\alpha}(c_i)$ has a minimum over $c_i \in [\underline{c},\bar{c}]$. Taking the derivative of U_i^{α} with respect to c_i and applying Lemma 1 yields the first-order necessary condition for an interior minimum:[6]

$$(\hat{C}) \qquad Q_i(\hat{c}_i) = Q^{\alpha}(\hat{c}_i).$$

The worst-off type must either be at an extreme point ($\underline{c}$ or $\bar{c}$) or at an interior point satisfying ($\hat{C}$). ■

Lemma 2 states that the individual-rationality constraint is binding at an extreme point ($\underline{c}$ or $\bar{c}$) or at an interior point $\hat{c}_i$ such that the firm's expected production under the cartel agreement is equal to the production without the cartel. Notice that a firm's net payoff $U_i^{\alpha}(\cdot) = \pi_i(\cdot) - \pi^{\alpha}(\cdot)$ is increasing whenever expected production with the cartel is less than production without the cartel.

We now provide a general characterization of enforceable cartels.

Theorem 1. *For any production rule $q \geq 0$, there exists a revenue rule r such that the cartel $\langle q,r\rangle$ is enforceable if and only if $Q_i(\cdot)$ is decreasing and for all $\hat{c}_i \in \hat{C}_i$*

$$(E) \qquad \sum_{i=1}^{n} \left[R_i(\bar{c}) - \bar{c}Q_i(\bar{c}) + \int_{\hat{c}_i}^{\bar{c}} Q_i(v)dv \right] \geq \sum_{i=1}^{n} \pi^{\alpha}(\hat{c}_i)$$

where

$$(R) \qquad \sum_{i=1}^{n} R_i(\bar{c}) = E[\rho(c)] + \sum_{i=1}^{n} \int_{\underline{c}}^{\bar{c}} vF(v)dQ_i(v).$$

Proof: *Only if.* Suppose $\langle q,r\rangle$ is incentive compatible and individually rational. Then from Lemma 1, for any c' and c_i

$$R_i(c') = R_i(c_i) - \int_{c'}^{c_i} vdQ_i(v).$$

Integrating over all types in $[\underline{c},\bar{c}]$ yields

[6] To simplify notation, we assume that both Q_i and Q^{α} are continuous.

$$E_i\{R_i(c')\} = R_i(c_i) - \int_{c'=\underline{c}}^{\bar{c}} \int_{v=c'}^{c_i} v dQ_i(v) dF(c')$$

$$= R_i(c_i) + \int_{v=c_i}^{\bar{c}} \int_{c'=v}^{\bar{c}} dF(c') v dQ_i(v) - \int_{v=\underline{c}}^{c_i} \int_{c'=\underline{c}}^{v} dF(c') v dQ_i(v)$$

$$= R_i(c_i) + \int_{c_i}^{\bar{c}} [1 - F(v)] v dQ_i(v) - \int_{\underline{c}}^{c_i} F(v) v dQ_i(v).$$

where the second line follows from changing the order of integration. Budget balance requires that

$$\sum_{i=1}^{n} r_i(c) = \rho(c) \qquad \text{for all } c \in [\underline{c},\bar{c}]^n,$$

so we have

$$\sum_{i=1}^{n} E_i\{r_i(c')\} = E\left\{ \sum_{i=1}^{n} r_i(c) \right\} = E[\rho(c)].$$

Therefore, summing over all firms yields

$$\text{(T)} \quad \sum_{i=1}^{n} R_i(c_i) = E[\rho(c)] + \sum_{i=1}^{n} \left[\int_{\underline{c}}^{c_i} F(v) v dQ_i(v) - \int_{c_i}^{\bar{c}} [1 - F(v)] v dQ_i(v) \right].$$

Evaluating (T) at $c_i = \bar{c}$ results in (R). Integrating (IC) by parts and letting $c' = \bar{c}$ and $c_i = \hat{c}_i$ yields

$$(\hat{T}) \qquad R_i(\hat{c}_i) = R_i(\bar{c}) - \bar{c}Q_i(\bar{c}) + \hat{c}_i Q_i(\hat{c}_i) + \int_{\hat{c}_i}^{\bar{c}} Q_i(v) dv.$$

From Lemma 2, $R_i(\hat{c}_i) - \hat{c}_i Q_i(\hat{c}_i) \geq \pi^{\alpha}(\hat{c}_i)$, which implies

$$\sum_{i=1}^{n} [R_i(\hat{c}_i) - \hat{c}_i Q_i(\hat{c}_i)] \geq \sum_{i=1}^{n} \pi^{\alpha}(\hat{c}_i).$$

Substituting $(\hat{T})$ into the expression above yields (E).

If. The proof is by construction. Let

$$r_i(c) = t_i + \rho_i(c) + \int_{\underline{c}}^{c_i} vdQ_i(v) - \frac{1}{n-1} \sum_{j \neq i} \int_{\underline{c}}^{c_j} vdQ_j(v),$$

where

$$\rho_i(c) = \frac{1}{n} \left[\rho(c) - E_{-i}[\rho(c)] + \frac{1}{n-1} \sum_{j \neq i} E_{-j}[\rho(c)] \right].$$

Then $\sum \rho_i(c) = \rho(c)$ and $E_{-i}[\rho_i(c)] = \frac{1}{n} E[\rho(c)]$. After changing the order of integration,

$$R_i(c_i) = t_i + \frac{1}{n} E[\rho(c)] + \int_{\underline{c}}^{c_i} vdQ_i(v) - \frac{1}{n-1} \sum_{j \neq i} \int_{\underline{c}}^{\bar{c}} [1 - F(v)]vdQ_j(v),$$

so Lemma 1 guarantees that $\langle q,r \rangle$ is incentive compatible. Finally, we have individual rationality if and only if $R_i(\hat{c}_i) - \hat{c}_i Q_i(\hat{c}_i) \geq \pi^{\alpha}(\hat{c}_i)$ by Lemma 2. Budget balance requires $\sum r_i(c) = \rho(c)$, which implies $\sum t_i = 0$. Our hypothesis is that (E) is satisfied, which is equivalent to $\sum [R_i(\hat{c}_i) - \hat{c}_i Q_i(\hat{c}_i)] \geq \sum \pi^{\alpha}(\hat{c}_i)$; hence, if we let

$$t_i = \frac{1}{n} \sum_{i=1}^{n} R_i(\hat{c}_i) - \int_{\underline{c}}^{\hat{c}_i} vdQ_i(v) + \frac{1}{n-1} \sum_{j \neq i} \int_{\underline{c}}^{\bar{c}} [1 - F(v)]vdQ_j(v),$$

then $R_i(\hat{c}_i) = \frac{1}{n} \sum R_i(\hat{c}_i) \geq \pi^{\alpha}(\hat{c}_i) + \hat{c}_i Q_i(\hat{c}_i)$. ∎

Theorem 1 can be stated much more simply if the cartel production is symmetric. A production rule q is *symmetric* if $Q_i(\cdot) = Q_j(\cdot) \equiv Q(\cdot)$ for all $i,j \in N$. Due to the symmetry of our model, nothing is gained by looking at asymmetric production rules.

Corollary 1. *For any symmetric production rule $q \geq 0$, there exists a revenue rule r such that the cartel $\langle q,r \rangle$ is enforceable if and only if $Q(\cdot)$ is decreasing and for all $\hat{c} \in \{\underline{c},\bar{c}\} \cup \{v \mid Q(v) = Q^{\alpha}(v)\}$*

$$(E') \qquad R(\bar{c}) - \bar{c}Q(\bar{c}) + \int_{\hat{c}}^{\bar{c}} Q(v)dv \geq \pi^{\alpha}(\hat{c}),$$

where

$$(R') \qquad R(\bar{c}) = \frac{1}{n} E[\rho(c)] + \int_{\underline{c}}^{\bar{c}} vF(v)dQ(v).$$

Proof: Given that $Q_i(\cdot) = Q_j(\cdot) \equiv Q(\cdot)$ for all $i,j \in N$, it immediately follows that (E′) and (R′) are equivalent to (E) and (R) of Theorem 1. ■

3. Cournot and Bertrand Competition

In this section, we determine the industry outcome if a cartel does not form. Two alternative outcomes are considered: Cournot competition (Lemma 3) and Bertrand competition (Lemma 4). Figure 1 illustrates the profit functions under both the Cournot (π^C) and monopoly (π^M) outcomes, as well as the net payoff $U^C = \pi^M - \pi^C$, for the case with four firms and uniform uncertainty ($F(v) = v$).

Lemma 3. *Firm i's quantity Q^C, profit π^C, and revenue R^C under Cournot competition are given by*

$$(QC) \qquad Q^C(c_i) = \begin{cases} 0 & \text{if } c_i \geq \tilde{c} \\ \frac{1}{2}(\tilde{c} - c_i) & \text{if } c_i < \tilde{c} \end{cases}$$

$$(\pi C) \qquad \pi^C(c_i) = \begin{cases} 0 & \text{if } c_i \geq \tilde{c} \\ \frac{1}{4}(\tilde{c} - c_i)^2 & \text{if } c_i < \tilde{c} \end{cases}$$

$$(RC) \qquad R^C(c_i) = \begin{cases} 0 & \text{if } c_i \geq \tilde{c} \\ \frac{1}{4}(\tilde{c}^2 - c_i{}^2) & \text{if } c_i < \tilde{c} \end{cases}$$

where $\tilde{c} = \bar{c}$ if $2(a - \bar{c}) \geq (n - 1)[\bar{c} - E_i(c_i)]$ and otherwise $\tilde{c}$ is defined implicitly (and uniquely) by the equation

$$(\tilde{C}) \qquad 2(a - \tilde{c}) = (n - 1) \int_{\underline{c}}^{\tilde{c}} F(v)dv.$$

Proof: In the Cournot game, firm i seeks to maximize its expected profit given the output decisions of the other firms. Firm i's profit $\pi^C(c_i)$ is given by

$$(\pi C') \qquad \pi^C(c_i) = (\tilde{c} - c_i - Q_i)Q_i$$

where $\tilde{c} = a - (n - 1)\overline{Q}$ and $\overline{Q}$ is the expected value of Q_i. Taking the derivative of the profit function with respect to Q_i yields the first-order condition

$$Q_i = \frac{1}{2}(\tilde{c} - c_i).$$

Checking the boundary conditions we see that firm i's output is given by

$$Q_i = Q^C(c_i) = \begin{cases} 0 & \text{if } c_i \geq \tilde{c} \\ \frac{1}{2}(\tilde{c} - c_i) & \text{if } c_i < \tilde{c}. \end{cases}$$

Substituting Q_i into $(\pi C')$ yields (πC). Moreover,

$$\overline{Q} = E(Q_i) = \frac{1}{2}\int_{\underline{c}}^{\tilde{c}} (\tilde{c} - v)dF(v) = \frac{1}{2}\int_{\underline{c}}^{\tilde{c}} F(v)dv.$$

Substituting $\overline{Q}$ into the definition of $\tilde{c}$ yields the implicit condition $(\tilde{C})$ for $\tilde{c}$, which determines a unique $\tilde{c}$ since the lefthand side of $(\tilde{C})$ is strictly decreasing, the righthand side is strictly increasing, and the lefthand side is greater (less) than the righthand side at $\tilde{c} = c$ $(\overline{c})$. The formula for (RC) follows from a simple calculation. ■

With Bertrand competition, since output is homogeneous, only the lowest cost firm will produce, since it will be able to undercut all other firms. We imagine the pricing game working in approximately the following way. Each firm posts a price. After seeing all other firms' prices, each firm can post a (lower) price if it wishes. The procedure stops when no firm lowers its price. The lowest-priced firm satisfies demand at its final posted price. This game is strategically similar to an English auction. Each firm will initially post its monopoly price, $(a + c_i)/2$, and from then on meet the lowest competing price until that price falls below c_i. Figure 2 shows a firm's expected profit under the Bertrand and monopoly outcomes for an example with four firms and uniform uncertainty. Notice that a firm's net payoff U^B is increasing in its cost, so that individual rationality needs to be checked only for a firm with cost $\underline{c}$.

Lemma 4. *Firm i's quantity Q^B, profit π^B, and revenue R^B under Bertrand competition are given by*

$$(QB) \qquad Q^B(c_i) = (a - c_i)G(c_i) - \int_{c_i}^{\frac{1}{2}(a+c_i)} G(v)dv$$

$$(\pi B) \qquad \pi^B(c_i) = \int_{c_i}^{\frac{1}{2}(a+c_i)} (a + c_i - 2v)G(v)dv$$

$$(RB) \qquad R^B(c_i) = c_i(a - c_i)G(c_i) + \int_{c_i}^{\frac{1}{2}(a+c_i)} (a - 2v)G(v)dv$$

where $G(v) = [1 - F(v)]^{n-1}$.

Proof: Let c_ℓ be the lowest cost among the n firms and let c_s be the second-lowest cost. Under Bertrand competition price falls to the second lowest cost, c_s, or the monopoly price of the lowest-cost firm, $\frac{1}{2}(a + c_\ell)$, whichever is less. Thus, firm i's ex post production is

$$q_i^B(c) = \begin{cases} 0 & \text{if } c_i > c_\ell \\ a - c_s & \text{if } c_i = c_\ell \text{ and } c_s \le \frac{1}{2}(a + c_\ell) \\ \frac{1}{2}(a - c_i) & \text{if } c_i = c_\ell \text{ and } c_s > \frac{1}{2}(a + c_\ell) \end{cases}$$

Firm i has the lowest cost if the cost of each of the other n - 1 firms is greater than c_i, so

$$\Pr\{\min_{i \neq j} c_j > c_i\} = [1 - F(c_i)]^{n-1} \equiv G(c_i).$$

The interim production for firm i is then found by integrating over the possible costs of the second-lowest cost firm:

$$Q^B(c_i) = -\int_{c_i}^{\frac{1}{2}(a+c_i)} (a - v)dG(v) - \int_{\frac{1}{2}(a+c_i)}^{\bar{c}} \frac{1}{2}(a - c_i)dG(v).$$

Performing the integration (by parts) results in

$$Q^B(c_i)=a[G(c_i)-G(\tfrac{1}{2}(a+c_i))] + vG(v)\Big|_{c_i}^{\frac{1}{2}(a+c_i)} - \int_{c_i}^{\frac{1}{2}(a+c_i)} G(v)dv + \frac{1}{2}(a-c_i)G(\tfrac{1}{2}(a+c_i)),$$

which after simplifying yields (QB). Similar calculations lead to the equations (πB) and (RB). ■

4. Ex Post Efficiency

A cartel $\langle q,r\rangle$ is *ex post efficient* if for each vector of costs c the outcome of the agreement $\{q(c),r(c)\}$ is Pareto-undominated by any alternative allocation, ignoring incentive constraints. Thus, ex post efficiency requires that the lowest-cost firm produces the monopoly output and the other firms produce nothing. In this section, we answer the question of whether this monopoly outcome can be attained by an enforceable cartel. We begin by determining the monopoly outcome.

Lemma 5. *Firm i's quantity Q^M, profit π^M, and revenue R^M in the monopoly outcome ignoring side-payments are given by*

(QM) $$Q^M(c_i) = \frac{1}{2}(a - c_i)G(c_i)$$

(πM) $$\pi^M(c_i) = \frac{1}{4}(a - c_i)^2G(c_i)$$

(RM) $$R^M(c_i) = \frac{1}{4}(a^2 - c_i^2)G(c_i)$$

where $G(v) = [1 - F(v)]^{n-1}$.

Proof: The monopoly output for the lowest-cost firm $c_\ell = \min c_i$ is given by

$$q_\ell(c_\ell) = \underset{q}{\text{argmax}}\ \{(a - q - c_\ell)q\} = \frac{1}{2}(a - c_\ell).$$

Hence, the ex post efficient production function is

$$q_i^M(c) = \begin{cases} \frac{1}{2}(a - c_\ell) & \text{if } c_i = c_\ell \\ 0 & \text{if } c_i > c_\ell, \end{cases}$$

and by independence, the expected production is

$$Q^M(c_i) = \frac{1}{2}(a - c_i)[1 - F(c_i)]^{n-1} = \frac{1}{2}(a - c_i)G(c_i).$$

The equations (πM) and (RM) follow immediately. ■

We now show that the individual rationality constraint is not binding at $\overline{c}$, so that individual rationality need only be checked at $\underline{c}$ and $\hat{c}$ defined by ($\hat{C}$). Let $R^M(\cdot)$ be the expected revenue after side-payments from the monopoly outcome.

Lemma 6. *In an ex post efficient cartel, the expected revenue of the highest-cost firm is given by*

$$(\overline{R}) \qquad R^M(\overline{c}) = \int_{\underline{c}}^{\overline{c}} \frac{1}{4}(a - v)^2(n - 1)f(v)F(v)[1 - F(v)]^{n-2}dv > 0.$$

Proof: From Corollary 1,

$$R^M(\overline{c}) = \frac{1}{n} E[\rho(c)] + \int_{\underline{c}}^{\overline{c}} vF(v)dQ^M(v).$$

Observe that

$$\frac{1}{n} E[\rho(c)] = \frac{1}{n} \int_{\underline{c}}^{\overline{c}} \frac{1}{4}(a - v)(a + v)nf(v)[1 - F(v)]^{n-1}dv,$$

and

$$\int_{\underline{c}}^{\overline{c}} vF(v)dQ^M(v) = vF(v)Q^M(v) \Big|_{\underline{c}}^{\overline{c}} - \int_{\underline{c}}^{\overline{c}} [F(v) + vf(v)]Q^M(v)dv$$

$$= 0 - \int_{\underline{c}}^{\overline{c}} [F(v) + vf(v)]\frac{1}{2}(a - v)[1 - F(v)]^{n-1}dv.$$

Therefore,

$$R^M(\bar{c}) = \int_{\underline{c}}^{\bar{c}} \left[\frac{1}{4}(a - v)(a + v)f(v) - [F(v) + vf(v)]\frac{1}{2}(a - v) \right][1 - F(v)]^{n-1}dv,$$

and so

$$(R'') \qquad R^M(\bar{c}) = \int_{\underline{c}}^{\bar{c}} \frac{1}{4}(a - v)^2 f(v)[1-F(v)]^{n-1}dv - \int_{\underline{c}}^{\bar{c}} \frac{1}{2}(a - v)F(v)[1-F(v)]^{n-1}dv.$$

Next, integrating the second term by parts yields

$$\int_{\underline{c}}^{\bar{c}} \frac{1}{2}(a - v)F(v)[1 - F(v)]^{n-1}dv = - \frac{1}{4}(a - v)^2 F(v)[1 - F(v)]^{n-1} \Bigg|_{\underline{c}}^{\bar{c}}$$

$$+ \int_{\underline{c}}^{\bar{c}} \frac{1}{4}(a - v)^2 \left[f(v)[1 - F(v)]^{n-1} - f(v)F(v)(n - 1)[1 - F(v)]^{n-2} \right] dv$$

$$= \int_{\underline{c}}^{\bar{c}} \frac{1}{4}(a - v)^2 f(v)[1-F(v)]^{n-1}dv - \int_{\underline{c}}^{\bar{c}} \frac{1}{4}(a - v)^2 f(v)F(v)(n-1)[1-F(v)]^{n-2}dv.$$

Substituting this back into (R"), we get ($\bar{R}$). ■

Equation ($\bar{R}$) has a nice interpretation. It says that the revenue, and hence the expected profit, to the highest-cost firm equals 1/n times the expected industry profit if the *second* most profitable firm were to produce its monopoly output (since $n(n - 1)f(v)F(v)[1 - F(v)]^{n-2}$ is the density of the second-order statistic). This suggests why it is difficult to enforce the monopoly outcome: the highest-cost firm must be paid a substantial amount of money to report its type truthfully, *even if* it would not produce without the cartel. As the number of firms grows, the amount paid to the highest-cost firm converges to an equal share of the industry profits when the lowest-cost firm produces the monopoly output.

The next two theorems give a necessary and sufficient condition for the attainment of the monopoly outcome by an enforceable cartel with either a Cournot or Bertrand threat. Simply put, the necessary and sufficient condition is that individual rationality must be satisfied for the worst-off type. With Cournot competition, this worst-off type must occur at an interior point $\hat{c}$ such that firm $\hat{c}$'s production is the same with or without the cartel. With Bertrand competition, the worst-off type is $\underline{c}$, since the monopoly output is less than the Bertrand output for all types of firms, and hence a firm's net profit $U^C(\cdot)$ is increasing.

Theorem 2. *Under a threat of Cournot competition, the monopoly outcome can be attained by an enforceable cartel if and only if for all $\hat{c}$ such that $Q^M(\hat{c}) = Q^C(\hat{c})$ we have*

$$(MC) \qquad R^M(\bar{c}) + \int_{\hat{c}}^{\bar{c}} Q^M(v)dv \geq \pi^C(\hat{c}).$$

Proof: With the monopoly outcome and a Cournot threat, (MC) is equivalent to (E′), since $Q^M(\bar{c}) = 0$. Hence, Theorem 2 is equivalent to Corollary 1, except that individual rationality is only checked at the interior point $\hat{c}$. It suffices to show that the worst-off type of firm cannot be at either extreme point $\underline{c}$ or $\bar{c}$. Lemma 6 proves that firm $\bar{c}$ is not the worst-off type. Since $Q^M(\underline{c}) = \frac{1}{2}(a - \underline{c})$ and $Q^C(\underline{c}) = \frac{1}{2}(\tilde{c} - \underline{c})$, $Q^M(\underline{c}) > Q^C(\underline{c})$ so the firm's net payoff $U^C(\cdot)$ is decreasing at $\underline{c}$. Thus, $\underline{c}$ is not the worst-off type. ■

Theorem 3. *Under a threat of Bertrand competition, the monopoly outcome can be attained by an enforceable cartel if and only if*

$$(MB) \qquad R^M(\bar{c}) + \int_{\underline{c}}^{\bar{c}} Q^M(v)dv \geq \pi^B(\underline{c}).$$

Proof: With the monopoly outcome and a Bertrand threat, (MB) is equivalent to (E′) evaluated at $\underline{c}$. Hence, it suffices to show that the worst-off type of firm is $\underline{c}$. A firm's net payoff $U^B(\cdot)$ is increasing if and only if $Q^M(c_i) \leq Q^B(c_i)$; that is,

$$\frac{1}{2}(a - c_i)G(c_i) \le (a - c_i)G(c_i) - \int_{c_i}^{\frac{1}{2}(a+c_i)} G(v)dv,$$

or equivalently,

$$\frac{1}{2}(a - c_i) \ge \int_{c_i}^{\frac{1}{2}(a+c_i)} \frac{G(v)}{G(c_i)} dv.$$

But this inequality is always satisfied, since $G(v)/G(c_i) \le 1$ for all $v \ge c_i$. Therefore, $U^B(\cdot)$ is everywhere increasing, so the worst-off type must be $\underline{c}$. ■

In determining what happens when the number of firms increases, it perhaps makes more sense to expand demand as n grows, so that if the number of firms doubles then industry demand doubles as well. In this case, the inverse demand is given by $P(q) = a - \frac{1}{n} \sum q_i$. It is easy to show that the effect of expanding demand as n increases is to increase profit, revenue, and production by a factor of n, due to the linearity of costs. Moreover, the values $\hat{c}$ and $\tilde{c}$ are the same as when demand is held constant. Thus, the set of enforceable cartels remains the same whether demand is held constant or expanded as n increases. That ex post efficiency is unattainable as n grows is a consequence of the heightened competition among firms, rather than dividing a pie of constant size into smaller pieces.

It has proven difficult to show in general that ex post efficiency is not obtainable with sufficiently many firms. In order to derive the result for Cournot competition we need to show that $\hat{c}$ converges to $\underline{c}$ much faster than c. This is difficult, since there is no explicit solution for $\tilde{c}$ and $\hat{c}$ in general. We can, however, demonstrate the result for both Cournot and Bertrand competition when costs are uniformly distributed. In addition, the next theorem provides a general result for the Bertrand threat, assuming the distribution F satisfies the condition below.

(F) There exists $c^* > \underline{c}$ such that for all $v \in [\underline{c}, c^*]$, $F(v) < 2(v-\underline{c})/(a-\underline{c})$.

Theorem 4. *If (F) is satisfied, then for sufficiently large n, the monopoly outcome is not enforceable by the Bertrand threat.*

Proof: From Lemma 6,

$$R^M(\bar{c}) = \int_{\underline{c}}^{\bar{c}} \frac{1}{4}(a-v)^2(n-1)f(v)F(v)[1-F(v)]^{n-2}dv$$

$$= \frac{1}{n}\left(\frac{a-c}{2}\right)^2 - \int_{\underline{c}}^{\bar{c}} \frac{1}{2}(a-v)\left[\frac{1}{n}[1-F(v)]^n + F(v)[1-F(v)]^{n-1}\right]dv,$$

and

$$\int_{\underline{c}}^{\bar{c}} Q(v)dv = \int_{\underline{c}}^{\bar{c}} \frac{1}{2}(a-v)[1-F(v)]^{n-1}dv,$$

so

$$R^M(\bar{c}) + \int_{\underline{c}}^{\bar{c}} Q(v)dv = \frac{1}{n}\left(\frac{a-\underline{c}}{2}\right)^2 + \frac{n-1}{n}\int_{\underline{c}}^{\bar{c}}\left(\frac{a-v}{2}\right)[1-F(v)]^n dv$$

$$= \left(\frac{a-\underline{c}}{2}\right)^2 - \frac{n-1}{n}\int_{\underline{c}}^{\bar{c}}\left(\frac{a-v}{2}\right)^2 g_n(v)dv,$$

where $g_n(v) = nf(v)[1-F(v)]^{n-1}$. In addition,

$$\pi^B(\underline{c}) = \int_{\underline{c}}^{(a+\underline{c})/2}(a+\underline{c}-2v)G_{n-1}(v)dv = \left(\frac{a-\underline{c}}{2}\right)^2 - \int_{\underline{c}}^{(a+\underline{c})/2}\left[\frac{a-v}{2} - \frac{v-\underline{c}}{2}\right]^2 g_{n-1}(v)dv$$

Therefore, from Theorem 3, the monopoly outcome is not enforceable with the Bertrand threat if and only if

$$\frac{n-1}{n}\int_{\underline{c}}^{\bar{c}}\left(\frac{a-v}{2}\right)^2 g_n(v)dv > \int_{\underline{c}}^{(a+\underline{c})/2}\left[\frac{a-v}{2} - \frac{v-\underline{c}}{2}\right]^2 g_{n-1}(v)dv,$$

or equivalently

$$\int_{\underline{c}}^{\bar{c}}\left(\frac{a-v}{2}\right)^2[1-F(v)]g_{n-1}(v)dv > \int_{\underline{c}}^{(a+\underline{c})/2}\left[\frac{a-v}{2} - \frac{v-\underline{c}}{2}\right]^2 g_{n-1}(v)dv,$$

which holds if

$$\int_{\underline{c}}^{(a+\underline{c})/2} \left(\frac{a-v}{2}\right)^2 [1-F(v)] g_{n-1}(v)dv > \int_{\underline{c}}^{(a+\underline{c})/2} \left[\frac{a-v}{2} - \frac{v-\underline{c}}{2}\right]^2 g_{n-1}(v)dv.$$

Since $a - v > v - \underline{c} > 0$ for $v \in [\underline{c}, \gamma]$ where $\gamma = (a+\underline{c})/2$, the last inequality holds if and only if

$$\int_{\underline{c}}^{\gamma} \left(\frac{a-v}{2}\right)^2 [1-F(v)] g_{n-1}(v)dv > \int_{\underline{c}}^{\gamma} \left(\frac{a-v}{2}\right)^2 g_{n-1}(v)dv - \int_{\underline{c}}^{\gamma} \left(\frac{v-\underline{c}}{2}\right)\left(\frac{2a+\underline{c}-3v}{2}\right) g_{n-1}(v)dv,$$

or equivalently

$$\int_{\underline{c}}^{\gamma} \left(\frac{a-v}{2}\right)^2 F(v) g_{n-1}(v)dv < \int_{\underline{c}}^{\gamma} \left(\frac{v-\underline{c}}{2}\right)\left(\frac{2a+\underline{c}-3v}{2}\right) g_{n-1}(v)dv,$$

which holds for large n if for v sufficiently close to $\underline{c}$,

$$\text{(D)} \qquad \left(\frac{a-v}{2}\right)^2 F(v) < \left(\frac{v-\underline{c}}{2}\right)\left(\frac{2a+\underline{c}-3v}{2}\right).$$

To see this, let

$$D(v) = \left(\frac{a-v}{2}\right)^2 F(v) - \left(\frac{v-\underline{c}}{2}\right)\left(\frac{2a+\underline{c}-3v}{2}\right).$$

The condition (D) amounts to saying that there exists $c^* > \underline{c}$ such that for all $v \in (\underline{c}, c^*]$, $D(v) < 0$. Denote $\epsilon \equiv \min_{v\in[c^*/2, c^*]} |D(v)|$ and let $M \equiv \max_{v\in(c^*, \gamma)} |D(v)|$. Then

$$\int_{\underline{c}}^{\gamma} D(v)g_{n-1}(v)dv = \int_{\underline{c}}^{c^*/2} D(v)g_{n-1}(v)dv + \int_{c^*/2}^{c^*} D(v)g_{n-1}(v)dv + \int_{c^*}^{\gamma} D(v)g_{n-1}(v)dv$$

$$< \int_{\underline{c}}^{c^*/2} D(v)g_{n-1}(v)dv - \epsilon\Big([1-F(c^*/2)]^{n-1} - [1-F(c^*)]^{n-1}\Big) + M[1-F(c^*)]^{n-1}$$

$$= \int_{\underline{c}}^{c^*/2} D(v)g_{n-1}(v)dv$$

$$- \Big([1-F(c^*/2)]^{n-1} - [1-F(c^*)]^{n-1}\Big)\left(\epsilon - M\frac{1}{[(1-F(c^*/2))(1-F(c^*))]^{n-1} - 1}\right)$$

< 0 for large n, since $F(c^*/2) < F(c^*)$ implies that

$$\lim_{n\to\infty} \frac{1}{[(1-F(c^*/2))(1-F(c^*))]^{n-1} - 1} = 0.$$

Since F is continuous, a sufficient condition for (D) to hold is (F).■

5. An Example with Uniform Costs

Suppose that a = 1 and the firms' costs are uniformly distributed on [0,1]. Then ($\tilde{C}$) can be written as

$$2(1 - \tilde{c}) = (n - 1)\int_0^{\tilde{c}} v dv,$$

which implies that $\tilde{c} = 2/(\sqrt{n} + 1)$. In addition, for an ex post efficient cartel, $Q^M(v) = \frac{1}{2}(1 - v)^n$, so that $\hat{c}$ is found by finding the first positive root of the equation

$$(1 - \hat{c})^n = \tilde{c} - \hat{c}.$$

The interim production with Bertrand competition is found by substituting a = 1 and F(v) = v into (QB) to yield

$$Q^B(c_i) = (1 - c_i)^n - \int_{c_i}^{\frac{1}{2}(1+c_i)} (1 - v)^{n-1} dv$$

$$= (1 - c_i)^n + \frac{1}{n}(1 - v)^n \Big|_{c_i}^{\frac{1}{2}(1+c_i)} = \beta(1 - c_i)^n$$

where $\beta = 1 - \frac{1}{n}(1 - 2^{-n})$.

Table 1 summarizes other formulas for the monopoly, Cournot, and Bertrand outcomes. For example, the ex ante industry profit with the Cournot outcome is given by

$$n\int_{\underline{c}}^{\tilde{c}} \pi^C(v) dF(v) = n\int_0^{\tilde{c}} \frac{1}{4}(\tilde{c} - v)^2 dv = n\tilde{c}^3/12,$$

and the industry profit with the monopoly outcome is

$$n \int_{\underline{c}}^{\bar{c}} \pi^M(v)\,dF(v) = \int_0^1 \frac{n}{4}(1 - v)^{n+1} dv = \frac{n}{4(n + 2)}\ .$$

The expected revenue to the highest-cost firm is found by integrating ($\bar{R}$) by parts:

$$(\bar{R}') \qquad R^M(1) = \int_0^1 \frac{1}{4}(n - 1)v(1 - v)^n dv$$

$$= \frac{1}{4}(n - 1)\left[-\frac{v(1 - v)^{n+1}}{n + 1}\Bigg|_0^1 + \int_0^1 \frac{(1 - v)^{n+1}}{n + 1}\, dv \right]$$

$$= \frac{1}{4}(n - 1)\left[-\frac{(1 - v)^{n+2}}{(n + 1)(n + 2)}\Bigg|_0^1 \right] = \frac{n - 1}{4(n + 1)(n + 2)}.$$

In addition,

$$(QM') \qquad \int_{\hat{c}}^{\bar{c}} Q^M(v)\,dv = \int_{\hat{c}}^1 \frac{1}{2}(1 - v)^n dv = \frac{(1 - \hat{c})^{n+1}}{2(n + 1)}\ .$$

By substituting these values into (MC), we get that the monopoly outcome is attainable by an enforceable cartel with a Cournot threat if and only if

$$(MC') \qquad \frac{n - 1}{4(n + 1)(n + 2)} + \frac{(1 - \hat{c})^{n+1}}{2(n + 1)} \geq \frac{1}{4}(\tilde{c} - \hat{c})^2,$$

where $\tilde{c} = 2/(\sqrt{n} + 1)$ and $\hat{c}$ is the first positive root of $(1 - \hat{c})^n = \tilde{c} - \hat{c}$.

Similarly, substituting ($\bar{R}'$) and (QM') into (MB) yields

$$\frac{n - 1}{4(n + 1)(n + 2)} + \frac{1}{2(n + 1)} \geq \frac{\beta}{n + 1}$$

or equivalently

$$(MB') \qquad \frac{n - 1}{2(n + 1)} \geq 1 - \frac{2}{n}\left(1 - \frac{1}{2^n} \right).$$

It is easy to check that (MB') is violated for all $n \geq 2$; thus, with a Bertrand threat, the monopoly outcome is never attainable.

Table 1. Formulas with a = 1 and F(v) = v (ignoring side-payments)

	Firm Interim Calculations		
Outcome	Monopoly	Cournot	Bertrand
Quantity	$\frac{1}{2}(1 - c_i)^n$	$\frac{1}{2}(\tilde{c} - c_i)$	$\beta(1 - c_i)^n$
Profit	$\frac{1}{4}(1 - c_i)^{n+1}$	$\frac{1}{4}(\tilde{c} - c_i)^2$	$\frac{\beta}{n + 1}(1 - c_i)^{n+1}$
Revenue	$\frac{1}{4}(1 + c_i)(1 - c_i)^n$	$\frac{1}{4}(\tilde{c}^2 - c_i{}^2)$	$\frac{\beta}{n + 1}(nc_i + 1)(1 - c_i)^n$
	Industry Ex Ante Calculations		
Outcome	Monopoly	Cournot	Bertrand
Quantity	$\frac{n}{2(n + 1)}$	$\frac{n}{4}\tilde{c}^2$	$\frac{n\beta}{n + 1}$
Profit	$\frac{n}{4(n + 2)}$	$\frac{n}{12}\tilde{c}_3$	$\frac{n\beta}{(n + 1)(n + 2)}$
Revenue	$\frac{n(n + 3)}{4(n + 1)(n + 2)}$	$\frac{n}{6}\tilde{c}^3$	$\frac{2n\beta}{(n + 1)(n + 2)}$

$\tilde{c} = 2/(\sqrt{n} + 1)$ $\qquad\qquad$ $\beta = 1 - (1 - 2^{-n})/n$

Table 2. Numerical Calculations with a = 1 and F(v) = v

n	2	3	4	5	6	7	8	∞
$\hat{c}$	.2200	.1793	.1537	.1357	.1221	.1115	.1029	0
$\tilde{c}$	.8284	.7321	.6667	.6180	.5798	.5486	.5224	0
Outcome	**Ex Ante Industry Quantity**							
Monopoly	.3333	.3750	.4000	.4167	.4286	.4375	.4444	.5
Cournot	.3431	.4019	.4444	.4775	.5042	.5267	.5458	1
Bertrand	.4167	.5313	.6125	.6719	.7165	.7510	.7782	1
Outcome	**Ex Ante Industry Profit**							
Monopoly	.1250	.1500	.1667	.1786	.1875	.1944	.2000	.25
Cournot	.0948	.0981	.0988	.0984	.0975	.0963	.0950	0
Bertrand	.1042	.1063	.1021	.0960	.0896	.0834	.0778	0
Outcome	**Ex Ante Industry Revenue**							
Monopoly	.2083	.2250	.2333	.2381	.2411	.2431	.2444	.25
Cournot	.1895	.1962	.1975	.1967	.1949	.1926	.1901	0
Bertrand	.2083	.2125	.2042	.1920	.1791	.1669	.1556	0
	Individual Rationality of Worst-Off Type							
$n[\pi^M - \pi^C]$	.0148	.0160	.0105	.0019	−.0081	−.0186	−.0292	−.75
$n[\pi^M - \pi^B]$	−.0417	−.0813	−.1125	−.1362	−.1540	−.1676	−.1782	−.25

Table 2 presents numerical calculations for several n. The last two rows in the table present the individual-rationality constraint for the worst-off type of firm with Cournot and Bertrand competition, respectively. The monopoly outcome is attainable by an enforceable cartel if and only if the difference between the expected profit with the ex post efficient cartel and the expected profit without the cartel is nonnegative. Thus, with a Bertrand threat, the monopoly outcome is never enforceable; whereas, with a Cournot threat, the monopoly outcome is enforceable if and only if $n \leq 5$.

The fact that enforcing the monopoly outcome with few firms is more difficult with a Bertrand threat than a Cournot threat follows from the fact that Bertrand competition is *less* competitive than Cournot competition when there are few firms with uncertainty about costs.

6. Enforceable Cartels with Common Value Uncertainty

Sections 2-5 presented results for a private value model in which firm types (costs) are independent. This section presents a related analysis for the case in which firms have a common (but uncertain) cost, c, and each receives a signal, s_i, about the cost. While the basic approach is quite similar, the problem facing the cartel is much different, and much easier in at least one important respect. In particular, to achieve the joint monopoly outcome it is no longer necessary for the cartel to identify the most efficient firm, since all firms are equally efficient. The only uncertainty is how much to produce in aggregate, so the cartel has an extra degree of freedom: how to share production. Since *total* industry profits are independent of the individual allocation of production levels, it may be possible to generate effective side payments simply by altering the assigned production allocations.

As before, we may think of a cartel as a pair of functions $\langle t(\cdot), q(\cdot)\rangle$ that maps each vector of firm types into a vector of revenue shares and output assignments, respectively. It is feasible if the revenue shares never exceed the total cartel revenues. However, because of the information structure, a firm's expected profit is not quite so easy to represent. Denote:

$$Q_i(c,s_i) = E_{-i}\{q_i(s_{-i},s_i)|c\}$$

$$T_i(c,s_i) = E_{-i}\{t_i(s_{-i},s_i)|c\}.$$

Then, if the "true" cost equals c, and a firm receives signal s_i, we have

$$\text{(IC)} \qquad U_i(c,s_i) = T_i(c,s_i) - c\, Q_i(c,s_i).$$

Expecting over c, after conditioning on s_i gives

$$\text{(IC}'\text{)} \qquad U_i(s_i) = T_i(s_i) - E_c\{cQ_i(c,s_i) \mid s_i\}$$

where $U_i(s_i) = E_c\{U_i(c,s_i) \mid s_i\}$ and $T_i(s_i) = E_c\{T_i(c,s_i) \mid s_i\}$. At this point we encounter some difficulty since (IC′) apparently does not produce a single "envelope" characterization of incentive compatibility, like the one obtained for the private values model in Lemma 1. A sufficient (but not necessary) condition for incentive compatibility can be obtained as in Lemma 1 by applying the usual techniques to (IC), for each realization of c, but this is not particularly useful.

Thus the revelation principle approach appears to have limited usefulness in this context, at least for the purpose of characterizing mechanisms.[7] Fortunately, this problem contains sufficient structure so that we can still establish several results relating to the question of enforcing the joint monopoly solution, without deriving a condition like (E). In particular, we provide a set of assumptions about the threat game which is

[7]A similar problem is encountered in pure exchange settings (e.g. auctions) with common value uncertainty.

sufficient to guarantee the ability of large cartels to successfully attain the monopoly solution. This contrasts sharply with the results for private costs.

To establish this result, we examine a simple cartel rule which is incentive compatible. We then specify conditions on the threat game such that the cartel rule will also be individually rational. The mechanism is the simplest imaginable one, which we call *equal-share information pooling*. Each firm is asked to report s_i. This determines an optimal (monopoly) aggregate cartel production level $\pi^M(s_1, \ldots, s_n)$. Each firm is then required to produce an equal share of this amount, and retains the revenues from selling the output it produces. Notice that this mechanism does not involve side-payments. We claim that for a wide variety of threat games, this cartel rule works for sufficiently large cartels. We present an example which demonstrates that it can fail for small cartels, and explain how the mechanism may need to be embellished when there are few firms.

The information structure is the following. The constant average cost of production c is the same for all firms, but is unknown at the time production takes place. Let $C = [\underline{c}, \overline{c}]$ be the set of possible costs. Given $c \in C$, each firm observes a signal s_i which is an independent draw from a common distribution, conditional on c, denoted $F(s_i | c)$. We assume that signals take on values in the unit interval and, for all $c \in C$, $F(0|c) = 0$, $F(1|c) = 1$ and F is continuous and strictly increasing on [0,1]. To complete the notation, let H(c) be the marginal distribution on C and $G(c|s_i)$ be the conditional distribution of c, given an observation $s_i \in [0,1]$.

Rather than specifying the exact structure of a threat game, α, we represent the threat game by the output functions induced by symmetric Bayesian equilibrium behavior of the firms under the incentives generated by the threat game.

Let $q^n\colon [0,1]^n \to \Re^n_+$. We say that q^n is an n-firm threat if there exists a threat game α with a symmetric Bayesian equilibrium which induces an output function $q^n(\cdot)$. Since we are interested in enforcing cartels with many firms, we will have to consider sequences $q = \{q^n\}_{n=2}^{\infty}$ with the property for each n, q^n is an *n-firm threat*. We call q a *threat*.

We say that equal share information pooling is *enforceable relative to threat q* if for sufficiently large n it is incentive compatible and individually rational relative to q^n for all s_i. Since it is trivially incentive compatible for all n, we need only prove that it is individually rational for all types when n is sufficiently large. Fixing q, use the following notation

$$\tilde{\pi}^n_i(s,c) = [a - c - \sum_{j=1}^{n} q^n_j(s)]q^n_i(s) \quad \text{for all } s \in [0,1]^n,\ c \in C$$

$$\pi^n(t) = \int_{\underline{c}}^{\overline{c}} \int_0^1 \cdots \int_0^1 \tilde{\pi}^n_i(s,c)dF(s_1|c)\ldots dF(s_{i-1}|c)dF(s_{i+1}|c)\ldots dF(s_n|c)dG(c|t)$$

$$\pi_M(t) = \int_{\underline{c}}^{\overline{c}} [\max\{0,(a-c)/2\}]^2 dG(c|t) \quad \text{for all } t \in [0,1]$$

$$M^n(\epsilon,t) = \{s \in [0,1] \mid \pi^n(s) \geq \epsilon\pi^n(t)\} \quad \text{for all } \epsilon > 0,\ t \in [0,1]$$

$$\overline{\pi}^n = n\int_{\underline{c}}^{\overline{c}} \int_0^1 \pi^n(t)dF(t|c)dH(c).$$

Thus, $\tilde{\pi}^n_i(\cdot,\cdot)$ is the ex post profit to firm i under q^n as a function of the true cost and any vector of signals; $\pi^n(\cdot)$ is the interim expected profit to a firm under q^n as a function of that firm's signal; $M^n(\epsilon,t)$ is the set of firm types whose interim profits exceed ϵ times the interim profit of a type t firm; $\overline{\pi}^n$ is the ex ante aggregate expected profits of the group of firms under q^n; $\pi_M(t)$ is a bit more difficult to interpret, but it is the interim expected profits of a monopolist who has observed signal $t \in [0,1]$, but who is permitted to produce *after* observing cost. For large n, this will approximate n times the interim cartel profits of a type t firm under the equal share information pooling mechanism.

We say that a threat q is *uniformly competitive* if

$$\lim_{n\to\infty} \bar{\pi}^n = 0 \tag{q1}$$

and for all $t \in [0,1]$ there exists $\epsilon, \gamma > 0$ such that for all c and n

$$\int_{M^n(\epsilon,t)} dF(s_i|c) > \gamma. \tag{q2}$$

The first part of this definition requires that aggregate ex ante profits under q converge to 0. The second part of the definition requires that if some firm is earning positive profits under q^n then a nontrivial set of other firms is also earning positive profits. Furthermore, as n gets large, this set of other firms grows at a rate on the order n.

Theorem 5. *If q is uniformly competitive, then equal-share information pooling is an enforceable cartel.*

Proof: We need to show that for large n, the cartel is individually rational relative to α for all types $t \in [0,1]$. First, note that $\pi_M(t) = 0$ implies that $q^n(t) = 0$ for all t, so the cartel is individually rational for these firms. Therefore, suppose that $\pi_M(t) > 0$. With information pooling, the interim joint monopoly profits, conditional on s, converge to $\pi_M(t)$. Therefore, individual rationality will be satisfied for t if, for sufficiently large n,

$$\lim_{n\to\infty} n\pi^n(t) < \pi_M(t) \ .$$

We next show that the left hand side of this inequality converges to 0 if q is uniformly competitive. From definitions,

$$\bar{\pi}^n = n \int_{\underline{c}}^{\bar{c}} \int_0^1 \pi^n(s)dF(s|c)dH(c)$$

$$\geq n \int_{\underline{c}}^{\bar{c}} \int_{M^n(\epsilon,t)} \pi^n(s)dF(s|c)dH(c)$$

$$\geq n \int_{\underline{c}}^{\bar{c}} \epsilon\ \pi^n(t) \int_{M^n(\epsilon,t)} dF(s|c)\ dH(c)$$

$$\geq n\epsilon\gamma\pi^n(t)$$

Hence, $n\pi^n(t) < \bar{\pi}^n/\epsilon\gamma$ for all n. Therefore, by (q1)

$$\lim_{n\to\infty} n\pi^n(t) = 0. \blacksquare$$

We illustrate the above result with the following example. Let α be the interim Cournot quantity game. Let there be two possible signals and two possible costs. Let $a = 1$, $C = \{0,3\}$ with $h(0) = 1/2$, $h(3) = 1/2$, $\delta \in [0,1]$, and

$$f(s_i|c=0) = \begin{cases} \delta & \text{if } s_i < 1/2 \\ 2-\delta & \text{if } s_i \geq 1/2 \end{cases} \qquad f(s_i|c=3) = \begin{cases} \delta^2 & \text{if } s_i < 1/2 \\ 2-\delta^2 & \text{if } s_i \geq 1/2. \end{cases}$$

Therefore for small δ, low signals are rare, quite optimistic, and highly informative, whereas high signals are common, mildly pessimistic, and uninformative.

With two firms it can be shown that for small δ, the following is an equilibrium:

$$q^2(s_i) = \begin{cases} \dfrac{(1+\delta)a - 3\delta}{2(1+\delta) + \delta(1-\delta)/2} & \text{if } s_i < 1/2 \\ 0 & \text{if } s_i \geq 1/2. \end{cases}$$

Since completely uninformed firms would produce 0, a low-signal firm is able to act as a monopolist in the low-cost state and produces a/2. Its expected profit is approximately $a^2/4$. It is *not* individually rational for a low signal type to participate in the equal share information pooling cartel, since such a firm would have to part with (almost) half of its (almost) monopoly profit.

With n firms, the unique equilibrium is

$$q^n(s_i) = \begin{cases} \dfrac{(1+\delta)a - 3\delta}{2(1+\delta) + (n-1)\delta(1+\delta^2)/2} & \text{if } s_i < 1/2 \\ 0 & \text{if } s_i \geq 1/2. \end{cases}$$

Expected profit equals $(q^n)^2$ which converges to 0 on the order of n^2, so $n\pi^n(t) \to 0$ for all t. Expected cartel profits for a low-signal firm converges to $(a^2/4)/(1+\delta) > 0$. Expected cartel profits for a high-signal form converges to $(a^2/4)(2-\delta)/(4-\delta-\delta^2) > 0$.

It is interesting to note that, while the equal sharing rule does not work for small n, a rule in which one's share is an explicit function of the vector of reported signals is individually rational and incentive compatible, at least for small n. Under this rule, firms who submit a low signal are asked to produce a large proportion of the output. In particular, the share equals 0 if a firm submits a high signal and 1/k otherwise, where k is the number of firms submitting a low signal. This suggests that the ability for a cartel to enforce the joint monopoly solution with only common value uncertainty is even more general than we have established here, where we have been limited to mechanisms in which shares are independent of reports.

7. Discussion

We have presented a simple model of cartel enforcement in an effort to determine how uncertainty about costs limits the power of a cartel to enforce desirable outcomes. Several strong assumptions have been made in order to keep the analysis manageable. In this section, we discuss the effects of alternative assumptions.

No Side-Payments. If side-payments are not allowed, then the revenue rule must equal the actual revenue for each firm: $r_i(c) = [a - \sum q_i(c)]q_i(c)$. With this restriction, the set of enforceable cartels is greatly diminished. Indeed, it is easy to show that with private cost uncertainty the monopoly outcome is not attainable, since every firm has an incentive to understate its cost, so as to increase its expected production. With common cost uncertainty, the equal-share information pooling rule still applies.

Durability. Suppose that each firm votes for or against the proposed cartel $\langle q,r \rangle$. If the cartel is not approved unanimously, then the competitive outcome as specified by the threat α results. In deriving the threat α, we have assumed that nothing is learned from the voting process. Alternatively, we may want to allow some inference in the face of disagreement.[8] In particular, it seems plausible that if a firm votes against the cartel, it did so because it gains the least from the proposed rules. We have shown that a low-cost firm benefits the least from an ex post efficient cartel, since it can do nearly as well by producing on its own. Thus, the other firms should infer that a negative vote comes from a low-cost firm. In our private valuation model, this inference has the effect of reducing everyone else's output, since they expect high output from the disagreeing firm. The net effect, then, is to weaken the threat of disagreement, and therefore to reduce the set of enforceable cartels. That is, we conjecture that if the monopoly outcome is not enforceable with a passive inference from disagreement, it is still not enforceable with a durability restriction.

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[8] This informal discussion is in the spirit of Holmstrom and Myerson [1983] and Crawford [1985], although it conforms with neither of their definitions of durability.

With a common value model, durability is likely to have the opposite effect. Firms infer that a firm voting against the cartel has a low signal, and therefore they expand their output, since it is likely that the true cost is lower than they originally suspected. This more aggressive response in the face of a negative vote strengthens the threat, and hence the set of enforceable cartels is expanded by durability arguments. Thus we would conjecture that the monopoly outcome will be enforceable with a durability restriction, whenever it is enforceable with passive inference.

Increasing Costs or Risk Aversion. With increasing costs or risk aversion, the characterization result in Theorem 1 becomes much more difficult, since linearity is lost. Some insights into analyzing this more complex problem may be gleaned from the literature on auction design with risk averse bidders, as in for example Maskin and Riley [1984b].

Regulated Cartel. The objective of a cartel in this paper is to maximize producer surplus. If instead the cartel is formed and regulated by the government, then it seems likely that both producer and consumer surplus would be given positive weight in the objective function. An analysis of this problem would extend the work of Baron and Myerson [1982] from one to several firms.

Ex Ante Efficient Cartels. For the private costs model we were able to characterize under what circumstances an enforceable cartel can achieve the monopoly outcome, but when the monopoly outcome is not attainable, how much collusion is possible? Moreover, what cartel rules generate the largest industry profits? Another interesting question is whether the set of enforceable cartels converge to the competitive outcome as the number of firms goes to infinity; that is, does the ability to collude vanish as the number of firms grows?

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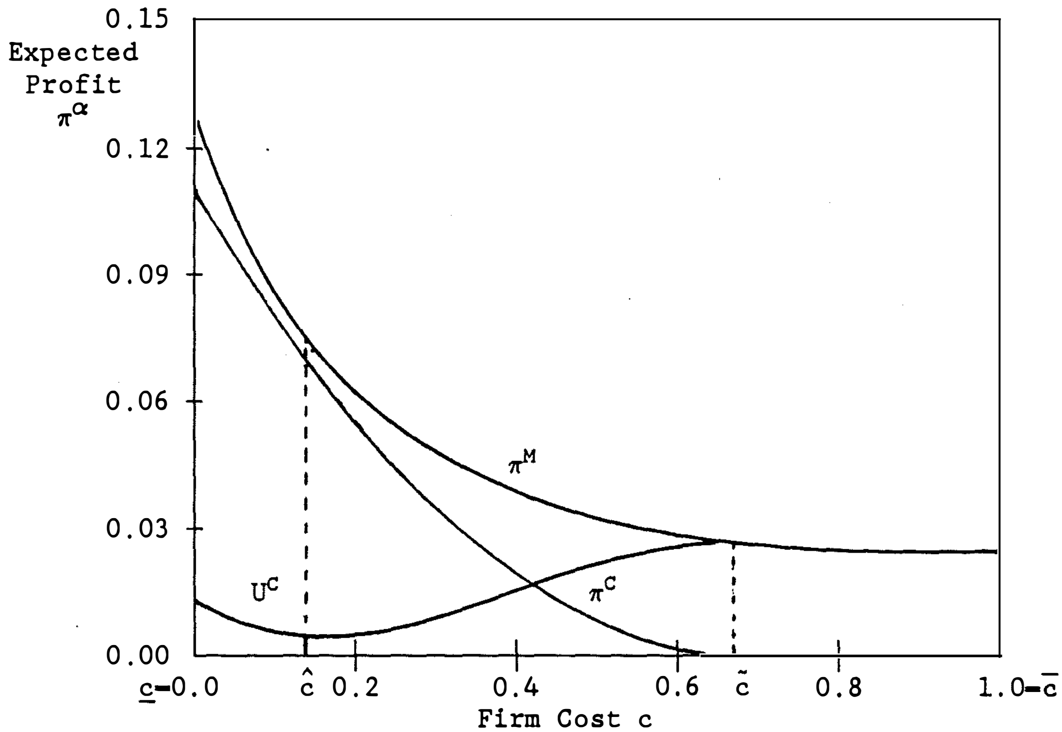

Figure 1. Expected Profit vs. Cost with a Cournot Threat (n = 4, Uniform)

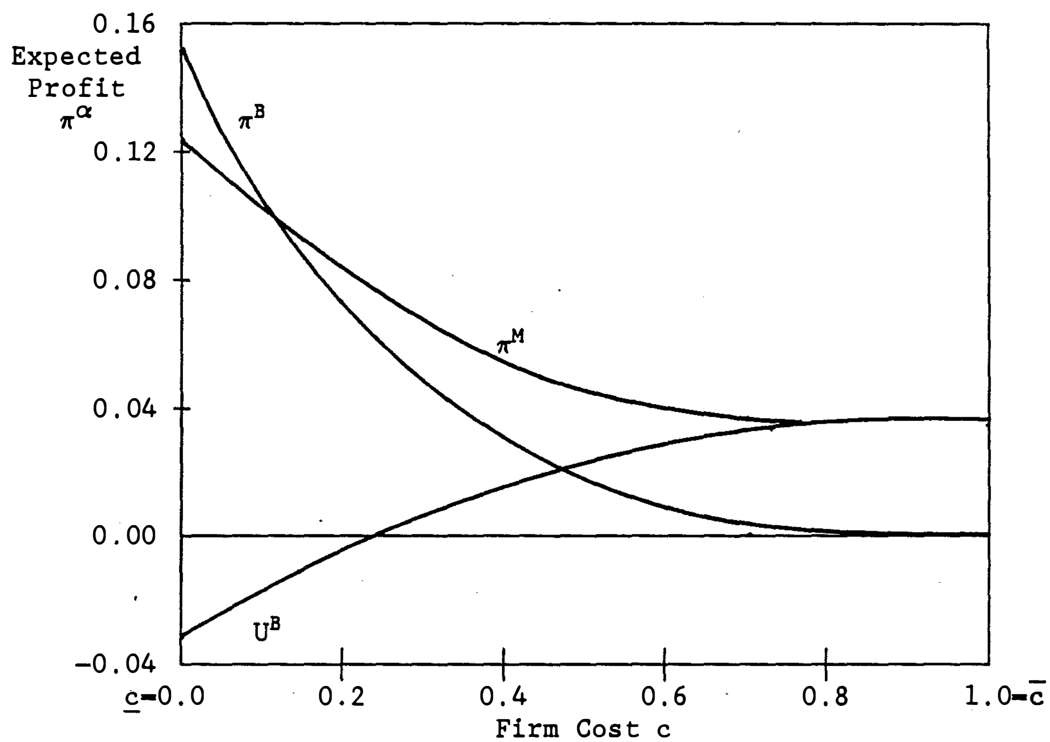

Figure 2. Expected Profit vs. Cost with a Bertrand Threat (n = 4, Uniform)